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ON THE PROJECTIVE DIFFERENTIAL GEOMETRY OF PLANE ANHARMONIC CURVES.

BY SAMUEL W. REAVES.

1. Introduction.—Following Halphen and Wilczynski[*] we shall interpret the fundamental solutions y_1, y_2, y_3 of a linear homogeneous differential equation of the third order

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0 \tag{1}$$

as the homogeneous coördinates of a point P_y. This point will describe an integral curve of (1) when the independent variable passes through all of its values. Associated with the point P_y are two other semi-covariant points[†] P_z, P_ρ, whose coördinates are given by the expressions

$$\begin{aligned} z_i &= y_i' + p_1y_i, \\ \rho_i &= y_i'' + 2p_1y_i' + p_2y_i \qquad (i = 1, 2, 3), \end{aligned} \tag{2}$$

and which, together with P_y, determine a triangle well suited to serve as triangle of reference for the purpose of investigating the properties of the curve in the vicinity of the point P_y. Of course as the independent variable changes these three points will trace three curves C_y, C_z, C_ρ.

Referred to the triangle $P_yP_zP_\rho$ the equation of the conic which osculates C_y at P_y is[‡]

$$x_2^2 - 2x_1x_3 + P_2x_3^2 = 0. \tag{3}$$

The covariant $\Theta_3'y + 3\Theta_3z$[§] determines a point P_γ whose coördinates are $(\Theta_3', 3\Theta_3, 0)$. It is the point in which the tangent at P_y meets the line upon which lie the three points of inflection of the eight-pointic nodal cubic of P_y.

The coördinates of the Halphen point P_h of P_y are[||]

$$\begin{aligned} x_1 &= 7(5\Theta_3'\Theta_8 - 756\Theta_3^4)^2 + 25\Theta_8^3 + 1575\Theta_3^2\Theta_8^2P_2, \\ x_2 &= 210\Theta_3\Theta_8(5\Theta_3'\Theta_8 - 756\Theta_3^4), \\ x_3 &= 3150\,\Theta_3^2\Theta_8^2. \end{aligned} \tag{4}$$

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[*] Projective Differential Geometry of Curves and Ruled Surfaces, B. G. Teubner, Leipzig, 1906, p. 60.

[†] Wilczynski, loc. cit., p. 61.

[‡] Wilczynski, loc. cit., p. 65.

[§] Wilczynski, loc. cit., pp. 85–86.

[||] Wilczynski, loc. cit., p. 68.

It is the purpose of this paper to find the equations of the loci C_z, C_ρ, C_γ of P_z, P_ρ, P_γ, the locus $C_{o'}$ of the center O' of the osculating conic, and the locus C_h of the Halphen point P_h, for each of the following anharmonic curves* (also known as W-curves):

$$(5) \qquad y = x^\lambda \qquad (\lambda \neq 0, 1, -1, \tfrac{1}{2}, 2),$$

$$(6) \qquad r = e^{m\theta},$$

$$(7) \qquad y = e^x.$$

For the logarithmic spirals (6), the osculating conic at an arbitrary point P_y will be studied in some detail and a construction for its center and axes will be given.

Let the sides of our triangle of reference be the line at infinity, the x-axis, and the y-axis, and let us choose the unit point in such a way that $(y_2/y_1, y_3/y_1)$ will be the cartesian coördinates of P_y. The relation between this system of coördinates and the local system determined by the semi-covariants y, z, ρ of the point P_y, is given by the equations

$$(8) \qquad \sigma_i = y_i x_1 + z_i x_2 + \rho_i x_3 \qquad (i = 1, 2, 3).$$

2. The Curves $y = x^\lambda$.—Since $x = y_2/y_1$ and $y = y_3/y_1$, this family of curves may be represented parametrically by the equations

$$(9) \qquad y_1 = 1, \qquad y_2 = t, \qquad y_3 = t^\lambda,$$

and since y_1, y_2, y_3 are to be solutions of a differential equation of the form (1), we find, on substitution, the following values for the coefficients:

$$(10) \qquad p_1 = \frac{2-\lambda}{3t}, \qquad p_2 = p_3 = 0.$$

Making use of equations (2) we find the coördinates of P_z, P_ρ, P_γ to be

$$(11) \qquad z_1 = \frac{2-\lambda}{3t}, \qquad z_2 = \frac{5-\lambda}{3}, \qquad z_3 = \frac{2\lambda+2}{3} t^{\lambda-1};$$

$$(12) \qquad \rho_1 = 0, \qquad \rho_2 = \frac{4-2\lambda}{3t}, \qquad \rho_3 = \frac{\lambda^2+\lambda}{3} t^{\lambda-2};$$

$$(13) \qquad \gamma_1 = \lambda + 1, \qquad \gamma_2 = (\lambda-2)t, \qquad \gamma_3 = (1-2\lambda)t^\lambda.$$

Placing in (11) $z_2/z_1 = x$, $z_3/z_1 = y$, and eliminating t, we have for the equation of C_z

$$(14) \qquad C_z: \quad y = \frac{2\lambda+2}{2-\lambda}\left(\frac{2-\lambda}{5-\lambda}\right)^\lambda x^\lambda.$$

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* Lie-Scheffers, Continuierliche Gruppen, pp. 68–82. Enc. der Math. Wiss., III, $D4$, pp. 204–215. Wilczynski, loc. cit., pp. 86–90.

In like manner we have from (12) and (13)

$$(15) \qquad C_\rho: \quad \text{The line at infinity, and}$$

$$(16) \qquad C_\gamma: \quad y = \frac{1-2\lambda}{1+\lambda}\left(\frac{\lambda+1}{\lambda-2}\right)^\lambda x^\lambda.$$

The equation of the osculating conic, as derived from (3) by means of the transformation (8), is

$$(17) \qquad \begin{aligned} &\lambda^2(2\lambda-1)(\lambda+1)t^{2\lambda}x^2 + 4\lambda(\lambda-2)(2\lambda-1)t^{\lambda+1}xy + (\lambda+1)(2-\lambda)t^2y^2 \\ &\quad + 4\lambda(\lambda^2-1)(2-\lambda)t^{2\lambda+1}x + 4(\lambda^2-1)(1-2\lambda)t^{\lambda+2}y \\ &\qquad + (\lambda-1)^2(\lambda-2)(2\lambda-1)t^{2\lambda+2} = 0. \end{aligned}$$

This conic is an hyperbola, parabola, or ellipse according as the expression $\lambda^2(\lambda-1)^2(\lambda-2)(2\lambda-1)t^{2\lambda+2}$ is positive, zero, or negative. The coördinates of the center O' are

$$(18) \qquad x = \frac{2\lambda+2}{2\lambda-1}t, \qquad y = \frac{2+2\lambda}{2-\lambda}t^\lambda,$$

and the locus $C_{o'}$ of the center has the equation

$$(19) \qquad C_{o'}: \quad y = \frac{2+2\lambda}{2-\lambda}\left(\frac{2\lambda-1}{2\lambda+2}\right)^\lambda x^\lambda.$$

Transforming the coördinates (4) of the Halphen point by means of (8), placing $\sigma_2/\sigma_1 = x$, $\sigma_3/\sigma_1 = y$, and eliminating t, we have for the equation of the locus C_h of P_h

$$(20) \qquad C_h: \quad y = \frac{A}{B}\left(\frac{B}{C}\right)^\lambda x^\lambda,$$

where

$$A = 27{,}876(\lambda^6+1) - 108{,}198(\lambda^5+\lambda) - 69{,}372(\lambda^4+\lambda^2) + 461{,}454\lambda^3,$$

$$B = 27{,}876\lambda^6 - 59{,}058\lambda^5 - 192{,}222\lambda^4 + 340{,}494\lambda^3 + 234{,}918\lambda^2 - 486{,}198\lambda + 162{,}066,$$

$$C = 162{,}066\lambda^6 - 486{,}198\lambda^5 + 234{,}918\lambda^4 + 340{,}494\lambda^3 - 192{,}222\lambda^2 - 59{,}058\lambda + 27{,}876.$$

It is thus seen that *the loci C_γ, $C_{o'}$ and C_h are all anharmonic curves with the same invariant and belonging to the same fundamental triangle as the original curve.*

3. The Logarithmic Spirals: $r = e^{m\theta}$.—The number m, which is the cotangent of the constant angle α which the tangent makes with the radius vector to the point of contact, may be called the *pitch* of the spiral. By putting

$$\frac{y_2}{y_1} = x = r\cos\theta$$

and

$$\frac{y_3}{y_1} = y = r\sin\theta$$

we may represent this family of spirals parametrically by the equations

$$(21) \qquad y_1 = 1, \qquad y_2 = e^{m\theta}\cos\theta, \qquad y_3 = e^{m\theta}\sin\theta.$$

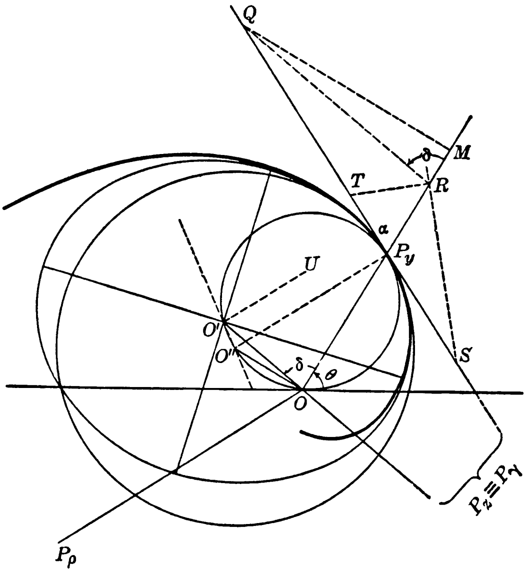
By methods similar to those of last paragraph we find values of $p_i, z_i, \rho_i, \gamma_i$ $(i = 1, 2, 3)$, and by eliminating θ we find the equations of the loci C_z, C_ρ, C_γ. It turns out that these are themselves logarithmic spirals with the same pole and pitch as the original spiral C_y, their equations being

$$(22) \qquad C_z \equiv C_\gamma: \quad r = \frac{\sqrt{m^2+9}}{2m}\,e^{m(\theta-\beta)},$$

$$(23) \qquad C_\rho: \quad r = \frac{2}{\sqrt{m^2+1}}\,e^{m\left(\theta-\alpha-\frac{\pi}{2}\right)},$$

where

$$\beta = \cot^{-1}\frac{m}{3}.$$



β and $(\pi/2 + \alpha)$ are the angles which OP_γ and OP_ρ make with OP_y, and these angles are therefore constant.

The equation of the osculating conic at P_y (r, θ) is

$$
\begin{aligned}
&[(17m^2+9)\cos^2\theta + (10m^3-6m)\sin\theta\cos\theta + (2m^4+3m^2+9)\sin^2\theta]x^2\\
&+2[(3m-5m^3)\cos^2\theta + (14m^2-2m^4)\sin\theta\cos\theta + (5m^3-3m)\sin^2\theta]xy\\
&+[(2m^4+3m^2+9)\cos^2\theta + (6m-10m^3)\sin\theta\cos\theta + (17m^2+9)\sin^2\theta]y^2\\
&+8[-2m^2\cos\theta + (3m+m^3)\sin\theta]e^{m\theta}x\\
&-8[(3m+m^3)\cos\theta + 2m^2\sin\theta]e^{m\theta}y - (m^2+9)e^{2m\theta} = 0.
\end{aligned}
\tag{24}
$$

The discriminant of the second degree terms, $ab - h^2$, has the value $9(m^2+1)^2(m^2+9)$, and since this is always positive *the osculating conic at all points of a logarithmic spiral is an ellipse.*

The coördinates of the center O' of this ellipse are

$$
x = \frac{4me^{m\theta}}{m^2+9}(m\cos\theta - 3\sin\theta), \qquad y = \frac{4me^{m\theta}}{m^2+9}(m\sin\theta + 3\cos\theta),
\tag{25}
$$

and the locus $C_{o'}$ of the center has the polar equation

$$
C_{o'}: \qquad r = \frac{4m}{\sqrt{m^2+9}}\, e^{m(\theta-\delta)}
\tag{26}
$$

where $\delta = \cot^{-1} m/3$ is the constant angle which OO' makes with OP_y. It may be shown that $\delta = \beta - \pi$.

The angle θ' which either axis of the osculating ellipse makes with the x-axis satisfies the equation

$$
\tan(2\theta' - 2\theta) = \frac{5m^2-3}{m^3-7m}.
\tag{27}
$$

Since $(\theta' - \theta)$ is constant, the axes of the ellipse make constant angles with the radius vector OP_y.

The lengths of the semi-axes of the osculating ellipse are found to be

$$
\begin{aligned}
a' &= \sqrt{2}r\sin\delta\,\frac{\sin\frac{\delta}{2}}{\sin\alpha}\\
b' &= \sqrt{2}r\sin\delta\,\frac{\cos\frac{\delta}{2}}{\sin\alpha}
\end{aligned}
\tag{28}
$$

and both axes are therefore proportional to the distance of the point of osculation from the pole. *Hence the osculating ellipse at all points of the logarithmic spiral remains similar to itself,* the constant eccentricity being $\sqrt{1-\tan^2\delta/2}$.

The center O'' of the osculating circle at P_y is found to have the coördinates

$$(29) \qquad x = -me^{m\theta}\sin\theta, \qquad y = me^{m\theta}\cos\theta,$$

and therefore OO'' is perpendicular to OP_y. Hence *the pole O lies on the circle having the radius of curvature $O''P_y$ for diameter*, and it may be shown that *the center O' of the osculating ellipse lies on the same circle.*

To construct the center O' of the osculating ellipse we may proceed as follows: From any point Q on the tangent at P_y drop a perpendicular to OP_y meeting it in M. Divide P_yM at R in the ratio $2:1$. Draw through O a line parallel to RQ to meet the circle $OO''P_y$ again in the required point O'.

The directions of the axes may be found by the following construction: Join $O'O''$ and draw $O'U$ parallel to $O''P_y$. The axes of the osculating ellipse will lie on the internal and external bisectors of the angle $O''O'U$. If $m > 0$, the major axis is the one which cuts OP_y on the side of O towards P_y.

Finally we may find the lengths of the axes geometrically as follows: If the internal and external bisectors of the angle QRM meet P_yQ in S and T, two sides of the triangle RP_yS have the ratio $\sin\delta/2 : \sin\alpha$, and two sides of the triangle RP_yT have the ratio $\cos\delta/2 : \sin\alpha$. If a perpendicular to OO' be dropped from P_y, its length is $r\sin\delta$. Hence, from (28), $a'/\sqrt{2}$ and $b'/\sqrt{2}$ may be found in magnitude by constructing fourth proportionals to these line segments, and then a' and b' may be readily found.

From the fact that the osculating ellipse has constant eccentricity, that its axes make constant angles with the radius vector OP_y, and that its center generates a logarithmic spiral having the same pole and pitch as the original spiral, it may be inferred that a great many other points of the figure will likewise generate spirals of the same pole and pitch. For example, this is true of the foci and extremities of the axes of the ellipse; also for the intersection points of the axes with OP_y, OP_ρ, OO'', $O''P_y$, QP_y, etc. And, dually, there are a great many lines of the figure which envelop spirals of the same pole and pitch as the original one. This is true, for example, of the axes of the ellipse, the lines joining the foci to the extremities of the minor axis, etc.

If we speak of all points and lines thus associated with a particular point P_y of the spiral, and which generate spirals while P_y traces the spiral C_y, as constituting a *set* of generators, we may say more generally, (a) any line through a point generator of a set, making a constant angle with OP_y, is a line generator of the same set; (b) a point dividing in a constant ratio the line segment determined by two point generators of the same set, is a point generator of the set; (c) the point of intersection of two line generators of the same set is a point generator of the set; and (*d*) the line joining two point generators of the same set is a line generator of the set.

The Halphen point and its locus may be found by the method of the last paragraph. We find

$$C_h: \qquad r = \frac{\sqrt{Q^2 + R^2}}{P} e^{m\left(\theta - \tan^{-1}\frac{R}{Q}\right)}, \tag{30}$$

where

$$P = 54{,}022m^6 + 497{,}106m^4 + 1{,}147{,}122m^2 + 109{,}350,$$

$$Q = 9{,}292m^6 + 313{,}776\,m^4 + 2{,}184{,}732m^2 - 145{,}800,$$

$$R = -\,16{,}380m^5 - 325{,}080m^3 + 1{,}122{,}660m,$$

and $\tan^{-1} R/Q$ is the constant angle which OP_h makes with OP_y.

We have now seen that *the loci* C_γ, $C_{o'}$, $C_{o''}$, C_h *are all spirals of the same pole and pitch as the spiral from which they are derived.*

4. **The Exponential Curves:** $y = e^x$.—We may use the parametric equations

$$y_1 = 1, \qquad y_2 = t, \qquad y_3 = e^t, \tag{31}$$

and by the methods of paragraph 2 we shall find

$$C_z \equiv C_\gamma: \qquad y = -\,2e^3 e^x \tag{32}$$

C_ρ: The line at infinity.

The equation of the osculating conic is

$$2e^{2t}x^2 + 8e^t xy + 7y^2 + (10 - 4t)e^{2t}x + (8 - 8t)e^t y + (2t^2 - 10t + 17) = 0, \tag{33}$$

and *it is therefore always an hyperbola.*

The locus of the center of the osculating conic and the locus of the Halphen point have respectively the equations

$$C_{o'}: \qquad y = -\,6e^{x - \frac{29}{2}}, \tag{34}$$

$$C_h: \qquad y = \frac{27511}{4646} e^{x + \frac{4095}{2323}}. \tag{35}$$

Thus *the loci* C_γ, $C_{o'}$, C_h *are all anharmonic curves of the same kind as the original curve* C_y.

UNIVERSITY OF OKLAHOMA,
September 14, 1912.